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CONSTITUTIVE EQUATIONS OF AN ISOTROPIC HYPERELASTIC BODY

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Stress–strain equations for an isotropic hyperelastic body are formulated. It is shown that the strain energy density whose gradient determines stresses can be defined as a function of two rather than three arguments, namely, strain-tensor invariants. In the case of small strains, the equations become relations of Hooke's law with two material constants, namely, shear modulus and bulk modulus.

1. Strains. We consider two interrelated (Cartesian and curvilinear) coordinate systems in the Euclidean space. We denote the Cartesian and curvilinear coordinates of a material point at the initial moment $\tau = 0$ by y^i and x^i, respectively, and those at the current moment τ by $\hat{y}^i$ and $\hat{x}^i$, respectively. The radius-vectors of material points vary from $\boldsymbol{R} = y^i \boldsymbol{k}_i$ at $\tau = 0$ to $\hat{\boldsymbol{R}} = \hat{y}^i \boldsymbol{k}_i$ at the moment τ ($\boldsymbol{k}_i$ are the basis vectors of the Cartesian coordinate system). For the initial position of the points, the basis vectors and metric tensor of the curvilinear coordinate system are $\boldsymbol{l}_i = \boldsymbol{R}_{,x^i} = y^n_{,x^i} \boldsymbol{k}_n$ and $g_{ij} = \boldsymbol{l}_i \cdot \boldsymbol{l}_j$, respectively, and, for the current position, they are $\hat{\boldsymbol{l}}_i = \hat{\boldsymbol{R}}_{,\hat{x}^i} = \hat{y}^n_{,\hat{x}^i} \boldsymbol{k}_n$ and $\hat{g}_{ij} = \hat{\boldsymbol{l}}_i \cdot \hat{\boldsymbol{l}}_j$, respectively. We write the displacement vector in the form $\boldsymbol{u} = \hat{\boldsymbol{R}} - \boldsymbol{R} = w^i \boldsymbol{k}_i = u^i \boldsymbol{l}_i = \hat{u}^i \hat{\boldsymbol{l}}_i$, where $w^n = u^i y^n_{,x^i} = \hat{u}^i \hat{y}^n_{,\hat{x}^i}$. Moreover, we have $\hat{\boldsymbol{R}}_{,x^i} = \boldsymbol{l}_i + u^n_{,i} \boldsymbol{l}_n$ and $\boldsymbol{R}_{,\hat{x}^i} = \hat{\boldsymbol{l}}_i - \hat{u}^n_{;i} \hat{\boldsymbol{l}}_n$. Here the subscripts and superscripts i, j, m, and n take the values 1, 2, and 3; summation from 1 to 3 is performed over repeated indices. The variables in the subscript after the comma denote partial differentiation. The subscript i after the comma denotes covariant differentiation with respect to x^i, and that after the semicolon denotes covariant differentiation with respect to $\hat{x}^i$. Covariant differentiation with respect to x^i and $\hat{x}^i$ is performed in the same curvilinear coordinate system, but the differentiated vectors and tensors are resolved into the different basis vectors $\boldsymbol{l}_i$ and $\hat{\boldsymbol{l}}_i$, respectively. The vectors $\hat{\boldsymbol{R}}_{,x^i}$ are the basis vectors in the comoving Lagrangian coordinate system. In quasistatic problems, any monotonically increasing loading parameter can be used as τ.

We consider the elementary material fibers in the initial $d\boldsymbol{R} = \boldsymbol{R}_{,x^i}\, dx^i$ and current $d\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}_{,x^i}\, dx^i$ states. The covariant components of the Green strain tensor are determined as coefficients in the expression for the difference between the squared initial and current lengths of these fibers $|d\hat{\boldsymbol{R}}|^2 - |d\boldsymbol{R}|^2 = 2e_{ij}\, dx^i\, dx^j$ by the formulas

$$e_{ij} = (\hat{y}^m_{,x^i} \hat{y}^m_{,x^j} - y^n_{,x^i} y^n_{,x^j})/2 = (\hat{g}_{mn} \hat{x}^m_{,x^i} \hat{x}^n_{,x^j} - g_{ij})/2 = (u_{i,j} + u_{j,i} + u^n_{,i}\, u_{n,j})/2. \tag{1.1}$$

Specifying the differentials of the current coordinates of the material points, we determine the fibers in the initial $d\boldsymbol{R}' = \boldsymbol{R}_{,\hat{x}^i}\, d\hat{x}^i$ and current $d\hat{\boldsymbol{R}}' = \hat{\boldsymbol{R}}_{,\hat{x}^i}\, d\hat{x}^i$ states. The coefficients in the expression $|d\hat{\boldsymbol{R}}'|^2 - |d\boldsymbol{R}'|^2 = 2\hat{e}_{ij}\, d\hat{x}^i\, d\hat{x}^j$ are the covariant components of the Almansi strain tensor

$$\hat{e}_{ij} = (\hat{y}^m_{,\hat{x}^i} \hat{y}^m_{,\hat{x}^j} - y^n_{,\hat{x}^i} y^n_{,\hat{x}^j})/2 = (\hat{g}_{ij} - g_{mn} x^m_{,\hat{x}^i} x^n_{,\hat{x}^j})/2 = (\hat{u}_{i;j} + \hat{u}_{j;i} - \hat{u}^n_{;i}\, \hat{u}_{n;j})/2. \tag{1.2}$$

The tensors $e = e_{ij}\, \boldsymbol{l}^i\, \boldsymbol{l}^j$ and $\hat{e} = \hat{e}_{ij}\, \hat{\boldsymbol{l}}^i\, \hat{\boldsymbol{l}}^j$ determine the strain at the same material point at the current moment. From (1.1) and (1.2), we obtain the relations

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$$\hat{e}_{mn} = e_{ij} x^i_{,\hat{x}^m} x^j_{,\hat{x}^n}, \qquad e_{ij} = \hat{e}_{mn} \hat{x}^m_{,x^i} \, \hat{x}^n_{,x^j}, \tag{1.3}$$

which relate the values of the covariant components of e and $\hat{e}$.

We introduce the matrices of partial derivatives $\hat{Y} = \|\hat{y}^i_{,y^j}\|$, $\hat{\Gamma} = \|\hat{x}^i_{,x^j}\|$, $\Pi = \|y^i_{,x^j}\|$, $\hat{\Pi} = \|\hat{y}^i_{,\hat{x}^j}\|$, $Y = \hat{Y}^{-1}$, and $\Gamma = \hat{\Gamma}^{-1}$ and the matrices of the covariant components of the strain and metric tensors $E = \|e_{ij}\|$, $\hat{E} = \|\hat{e}_{ij}\|$, $G = \|g_{ij}\|$, and $\hat{G} = \|\hat{g}_{ij}\|$ (i and j are the row and column numbers, respectively). In matrix notation, we obtain

$$E = (\hat{\Gamma}^{\mathrm{t}} \hat{G} \hat{\Gamma} - G)/2, \qquad \hat{E} = (\hat{G} - \Gamma^{\mathrm{t}} G \Gamma)/2, \tag{1.4}$$

and, hence, $\hat{E} = \Gamma^{\mathrm{t}} E \, \Gamma$ (the superscript "t" denotes the transpose of matrices).

To determine the volume strain, we use the Cartesian coordinate system. For an undeformed body, we consider an elementary parallelepiped bounded by the coordinate planes and having edges of length dy^i and volume $dV_0 = dy^1 dy^2 dy^3$. At the current moment, it occupies the volume $dV = (\hat{\boldsymbol{R}}_{,y^1} \times \hat{\boldsymbol{R}}_{,y^2}) \cdot \hat{\boldsymbol{R}}_{,y^3} \, dV_0 = J \, dV_0$, where $J = \det \hat{Y}$ is the Jacobian of transformation of the initial coordinates to the current coordinates of the material points. The volume strain ε_V is determined by the formulas $J = dV/dV_0$ and $\varepsilon_V = (dV - dV_0)/dV_0 = J - 1$. Equating the matrix determinants in the equalities $\hat{Y}^{\mathrm{t}} \hat{Y} = (\Pi^{-1})^{\mathrm{t}} (G + 2E) \Pi^{-1}$ and $Y^{\mathrm{t}} Y = (\hat{\Pi}^{-1})^{\mathrm{t}} (\hat{G} - 2\hat{E}) \hat{\Pi}^{-1}$ implied by (1.4), we express the Jacobian in terms of the components of the strain tensors e and $\hat{e}$:

$$J = (\det G)^{-1/2} [\det(G + 2E)]^{1/2} = (\det \hat{G})^{1/2} [\det(\hat{G} - 2\hat{E})]^{-1/2}. \tag{1.5}$$

2. Strain Rates. Let $\dot{\boldsymbol{u}} = \boldsymbol{v} = v^i \boldsymbol{l}_i = \hat{v}^i \hat{\boldsymbol{l}}_i$ be the velocities of the material points at the moment τ (the dot denotes differentiation with respect to τ). We assume that, at the moment τ, the radius-vectors of the points ($\hat{\boldsymbol{R}}$) and the elementary material fibers determined by the differentials dx^i and $d\hat{x}^i$ ($d\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}_{,x^i} \, dx^i$ and $d\hat{\boldsymbol{R}}' = \hat{\boldsymbol{R}}_{,\hat{x}^i} \, d\hat{x}^i$, respectively) change to $\hat{\boldsymbol{R}}_1 = \hat{\boldsymbol{R}} + \boldsymbol{v} \Delta\tau$, $d\hat{\boldsymbol{R}}_1 = \hat{\boldsymbol{R}}_{1,x^i} \, dx^i$, and $d\hat{\boldsymbol{R}}'_1 = \hat{\boldsymbol{R}}_{1,\hat{x}^i} \, d\hat{x}^i$, respectively, at the moment $\tau + \Delta\tau$ ($\Delta\tau$ is the small period). Determining the rates of variation of the squared lengths of these fibers

$$\lim_{\Delta\tau \to 0} \frac{|d\hat{\boldsymbol{R}}_1|^2 - |d\hat{\boldsymbol{R}}|^2}{\Delta\tau} = 2\eta_{ij} \, dx^i \, dx^j, \qquad \lim_{\Delta\tau \to 0} \frac{|d\hat{\boldsymbol{R}}'_1|^2 - |d\hat{\boldsymbol{R}}'|^2}{\Delta\tau} = 2\hat{\eta}_{ij} \, d\hat{x}^i \, d\hat{x}^j,$$

we obtain the covariant components of the two strain-rate tensors

$$\eta_{ij} = \dot{e}_{ij} = (\boldsymbol{v}_{,x^i} \cdot \hat{\boldsymbol{R}}_{,x^j} + \boldsymbol{v}_{,x^j} \cdot \hat{\boldsymbol{R}}_{,x^i})/2 = (v_{i,j} + v_{j,i} + u^n_{,i} \, v_{n,j} + u^n_{,j} \, v_{n,i})/2,$$

$$\hat{\eta}_{ij} = (\boldsymbol{v}_{,\hat{x}^i} \cdot \hat{\boldsymbol{R}}_{,\hat{x}^j} + \boldsymbol{v}_{,\hat{x}^j} \cdot \hat{\boldsymbol{R}}_{,\hat{x}^i})/2 = (\hat{v}_{i;j} + \hat{v}_{j;i})/2,$$

which are related by

$$\hat{\eta}_{mn} = \eta_{ij} \, x^i_{,\hat{x}^m} x^j_{,\hat{x}^n}, \qquad \eta_{ij} = \hat{\eta}_{mn} \, \hat{x}^m_{,x^i} \, \hat{x}^n_{,x^j}. \tag{2.1}$$

The tensors $\eta = \eta_{ij} \, \boldsymbol{l}^i \boldsymbol{l}^j$ and $\hat{\eta} = \hat{\eta}_{ij} \, \hat{\boldsymbol{l}}^i \, \hat{\boldsymbol{l}}^j$ are the strain rates at the current moment at the same material point, but they correspond to different states of the body (the tensors η and $\hat{\eta}$ refer to the initial and current states, respectively). The tensor η is the Green strain-rate tensor $\eta = \dot{e}$.

Differentiating the Jacobian $J = \det \hat{Y}$ with respect to τ, we obtain $\dot{J} = J \dot{w}^m_{,\hat{y}^m}$. The equalities $\boldsymbol{v}_{,\hat{x}^i} = \dot{w}^m_{,\hat{x}^i} \boldsymbol{k}_m = \hat{v}^n_{;i} \hat{\boldsymbol{l}}_n$ imply that $\dot{w}^m_{,\hat{x}^i} = \hat{v}^n_{;i} \, \hat{y}^m_{,\hat{x}^n}$, $\dot{w}^m_{,\hat{y}^n} = \hat{v}^j_{;i} \, \hat{y}^m_{,\hat{x}^j} \, \hat{x}^i_{,\hat{y}^n}$, and $\dot{w}^m_{,\hat{y}^m} = \hat{v}^i_{;i} = \hat{\eta}^i_i$. The volume-strain rate is given by $\dot{\varepsilon}_V = \dot{J} = J \, \hat{\eta}^i_i$. Hence, $\hat{\eta}^i_i = \dot{J}/J$.

We decompose $\hat{\eta}$ into the deviatoric and spherical parts:

$$\hat{\eta}_{ij} = \hat{\eta}^{(1)}_{ij} + \hat{\eta}^{(2)}_{ij}, \qquad \hat{\eta}^{(1)}_{ij} = \hat{\eta}_{ij} - \hat{\eta}^n_n \, \hat{g}_{ij}/3, \qquad \hat{\eta}^{(2)}_{ij} = \hat{\eta}^n_n \, \hat{g}_{ij}/3$$

and, according to (2.1), we find the decomposition of η:

$$\eta_{ij} = \eta^{(1)}_{ij} + \eta^{(2)}_{ij}, \qquad \eta^{(1)}_{ij} = \hat{\eta}^{(1)}_{mn} \, \hat{x}^m_{,x^i} \, \hat{x}^n_{,x^j} = \dot{e}_{ij} - (\dot{J}/(3J))(g_{ij} + 2e_{ij}) = J^{2/3} \dot{\theta}_{ij},$$
$$\eta^{(2)}_{ij} = \hat{\eta}^{(2)}_{mn} \, \hat{x}^m_{,x^i} \, \hat{x}^n_{,x^j} = \hat{\eta}^n_n \, \hat{y}^m_{,x^i} \hat{y}^m_{,x^j}/3. \tag{2.2}$$

The tensors $\hat{\eta}^{(1)} = \hat{\eta}^{(1)}_{ij}\,\hat{\boldsymbol{l}}^i\,\hat{\boldsymbol{l}}^j$, $\hat{\eta}^{(2)} = \hat{\eta}^{(2)}_{ij}\,\hat{\boldsymbol{l}}^i\,\hat{\boldsymbol{l}}^j$, $\eta^{(1)} = \eta^{(1)}_{ij}\,\boldsymbol{l}^i\,\boldsymbol{l}^j$, and $\eta^{(2)} = \eta^{(2)}_{ij}\,\boldsymbol{l}^i\,\boldsymbol{l}^j$ are symmetric. We have $\hat{\eta} = \hat{\eta}^{(1)} + \hat{\eta}^{(2)}$, $\eta = \eta^{(1)} + \eta^{(2)}$, and $\eta^{(1)} = J^{2/3}\dot{\theta}$. The tensor $\eta^{(1)}$ can be written in the form of a product of $J^{2/3}$ and the rate of variation of the tensor $\theta = \theta_{ij}\,\boldsymbol{l}^i\,\boldsymbol{l}^j$, whose components are expressed with allowance for (1.5) in terms of the components of the strain tensor e:

$$\theta_{ij} = J^{-2/3}(g_{ij} + 2e_{ij})/2 = J^{-2/3}\hat{y}^m_{,x^i}\hat{y}^m_{,x^j}/2. \tag{2.3}$$

The tensors e and θ are coaxial.

The determinant of the matrix $F = \|\theta^i_j\|$ composed of the mixed components of θ

$$F = J^{-2/3}G^{-1}(G + 2E)/2 = J^{-2/3}G^{-1}\Pi^{\mathrm{t}}\hat{Y}^{\mathrm{t}}\hat{Y}\Pi/2$$

is a constant that does not depend on strains:

$$\det F = J^{-2}(\det G)^{-1}(\det \Pi)^2(\det \hat{Y})^2/8 = J^{-2}(\det \hat{Y})^2/8 = 1/8.$$

This imposes a constraint on admissible values of the θ components, which cannot be arbitrary and independent of each other. In contrast to the tensor θ, the tensor e has not three but only two basis invariants that are variable and independent of the orientation of its principal axes. As these basis invariants, one can take

$$\Theta_1 = \theta^n_n = J^{-2/3}\,I_1, \qquad \Theta_2 = \theta^{ij\,\prime}\,\theta'_{ij} = J^{-4/3}\,I_2, \tag{2.4}$$

where $I_1 = 1.5 + e^n_n$ and $I_2 = e^{ij\,\prime}e'_{ij}$ are the invariants of e. The Jacobian J can be considered as an additional parameter which, together with θ, determines the strain tensor $e_{ij} = J^{2/3}\theta_{ij} - 0.5g_{ij}$. The deviatoric tensors $\theta'_{ij} = \theta_{ij} - (\theta^n_n/3)g_{ij} = J^{-2/3}e'_{ij}$ and $e'_{ij} = e_{ij} - (e^n_n/3)g_{ij}$, which correspond to e and θ, differ by the multiplier $J^{-2/3}$.

3. Stresses. We introduce the symmetric Cauchy and Piola–Kirchhoff stress tensors ($\hat{\sigma}$ and σ, respectively), which are conjugate with the tensors $\hat{\eta}$ and η, respectively. The double convolutions of the stress tensors with $\hat{\eta}$ and η are the power densities of stress work in the strain rates per unit volume in the current and initial states

$$\hat{\psi} = \hat{\sigma}^{ij}\hat{\eta}_{ij} = \hat{\sigma}^{ij}\hat{v}_{j;i}, \qquad \psi = \sigma^{ij}\eta_{ij} = \Sigma^{ij}v_{j,i}, \qquad \Sigma^{ij} = \sigma^{ij} + \sigma^{in}u^j_{,n}. \tag{3.1}$$

The tensors $\hat{\sigma} = \hat{\sigma}^{ij}\,\hat{\boldsymbol{l}}_i\,\hat{\boldsymbol{l}}_j$ and $\sigma = \sigma^{ij}\,\boldsymbol{l}_i\,\boldsymbol{l}_j$ determine the stress state at the same material point. The quantities Σ^{ij} in (3.1) are the contravariant components of the first nonsymmetric Piola–Kirchhoff stress tensor. Using the equalities $\psi = J\hat{\psi}$ and (2.1), we obtain the following relations between the contravariant components of the tensors σ and $\hat{\sigma}$:

$$\sigma^{ij} = J\hat{\sigma}^{mn}x^i_{,\hat{x}^m}x^j_{,\hat{x}^n}, \qquad \hat{\sigma}^{mn} = J^{-1}\sigma^{ij}\hat{x}^m_{,x^i}\hat{x}^n_{,x^j}. \tag{3.2}$$

We decompose $\hat{\sigma}$ into the deviatoric and spherical parts:

$$\hat{\sigma}^{ij} = \hat{\sigma}^{(1)ij} + \hat{\sigma}^{(2)ij}, \qquad \hat{\sigma}^{(1)ij} = \hat{\sigma}^{ij} - p\hat{g}^{ij}, \qquad \hat{\sigma}^{(2)ij} = p\hat{g}^{ij},$$

where $p = \hat{\sigma}^n_n/3$ is the average stress, which is called the hydrostatic pressure. According to (3.2), we obtain the decomposition of σ:

$$\sigma^{ij} = \sigma^{(1)ij} + \sigma^{(2)ij}, \qquad \sigma^{(1)ij} = J\hat{\sigma}^{(1)mn}x^i_{,\hat{x}^m}x^j_{,\hat{x}^n}, \qquad \sigma^{(2)ij} = pJ\alpha^{ij},$$
$$\alpha^{ij} = x^i_{,\hat{y}^m}x^j_{,\hat{y}^m} = [(G + 2E)^{-1}]^{ij}. \tag{3.3}$$

The tensors $\hat{\sigma}^{(1)} = \hat{\sigma}^{(1)ij}\hat{\boldsymbol{l}}_i\hat{\boldsymbol{l}}_j$, $\hat{\sigma}^{(2)} = \hat{\sigma}^{(2)ij}\hat{\boldsymbol{l}}_i\hat{\boldsymbol{l}}_j$, $\sigma^{(1)} = \sigma^{(1)ij}\boldsymbol{l}_i\boldsymbol{l}_j$, $\sigma^{(2)} = \sigma^{(2)ij}\boldsymbol{l}_i\boldsymbol{l}_j$, and $\alpha = \alpha^{ij}\,\boldsymbol{l}_i\,\boldsymbol{l}_j$ are symmetric. We have $\hat{\sigma} = \hat{\sigma}^{(1)} + \hat{\sigma}^{(2)}$, $\sigma = \sigma^{(1)} + \sigma^{(2)}$, and $\sigma^{(2)} = pJ\alpha$.

Using (2.2) and (3.3), and the equalities $\sigma^{(1)ij}\eta^{(2)}_{ij} = \sigma^{(2)ij}\eta^{(1)}_{ij} = 0$, we write the power density of stress work in the strain rates in the form

$$\psi = s^{ij}\dot{\theta}_{ij} + p\dot{J}, \tag{3.4}$$

where

$$s^{ij} = J^{2/3}\sigma^{(1)ij}. \tag{3.5}$$

Substituting the expressions

$$\hat{\sigma}^{(1)mn} = J^{-5/3}s^{ij}\,\hat{x}^m_{,x^i}\hat{x}^n_{,x^j}, \qquad \hat{\eta}^{(1)}_{mn} = J^{2/3}\dot{\theta}_{ij}\,x^i_{,\hat{x}^m}x^j_{,\hat{x}^n}$$

implied by (2.1), (2.2), (3.3), and (3.5) into the equations $\hat{\sigma}^{(1)mn}\hat{g}_{mn} = \hat{\eta}^{(1)}_{mn}\hat{g}^{mn} = 0$, we obtain the equalities

$$s^{ij}\theta_{ij} = 0, \qquad \sigma^{(2)ij}\dot{\theta}_{ij} = 0, \tag{3.6}$$

which are used below to construct equations of an isotropic hyperelastic body. We note that, given p and s^{ij}, one can determine the stresses $\sigma^{ij} = J^{-2/3}s^{ij} + pJ\alpha^{ij}$ and, hence [using Eqs. (3.2)], $\hat{\sigma}^{ij}$.

4. Equations of an Isotropic Hyperelastic Body. In reversible thermodynamic processes [1], for each elementary material particle, for the small period $d\tau$, the increment of total energy dU takes the value

$$dU = dW + dQ + dA = (\boldsymbol{v}\cdot\dot{\boldsymbol{v}} + T\dot{S} + \rho_0^{-1}\psi)\,dM\,d\tau,$$

where dW is the kinetic-energy increment, dQ is the heat inflow, dA is the increment of stress work due to the action of the external mass and surface forces on the body ($\rho_0^{-1}\psi = \rho^{-1}\hat{\psi}$), T is the absolute temperature, S is the entropy density per unit mass of the body, $dM = \rho_0 dV_0 = \rho dV$ is the particle mass, and ρ_0 and ρ are the densities of material in the initial and current states, respectively. We confine our analysis to processes that are either isentropic when $\dot{S} = 0$ and, hence, adiabatic or isothermal processes. Consequently, in addition to dU, the increment of stress work dA must also be a differential. Satisfying this condition and taking (3.4) into account, we assume, in accordance with the definition of an isotropic hyperelastic body [2], that the quantity

$$d\Psi = \psi\,d\tau = s^{ij}\,d\theta_{ij} + p\,dJ \tag{4.1}$$

is the total differential of the strain-energy density per unit volume of the undeformed body $\Psi = \Psi(\Theta_1, \Theta_2, J)$ which depends on the Jacobian J and the invariants of the tensor θ (2.4). In this case, the work of stresses in each elementary material particle is equal to zero on any closed deformation path.

In accordance with (4.1), we assume

$$s^{ij} = \Psi_{,\theta_{ij}} = \Psi_{,\Theta_1}g^{ij} + 2\,\Psi_{,\Theta_2}\theta^{ij\,\prime}, \qquad p = \Psi_{,J}. \tag{4.2}$$

In the expression for s^{ij}, terms with $\sigma^{(2)ij}$ are ignored, since they do not contribute to the work of stresses in $\dot{\theta}_{ij}$ [see (3.6)]. Satisfying the first equality in (3.6), we obtain the first-order partial differential equation for Ψ

$$\Theta_1\Psi_{,\Theta_1} + 2\Theta_2\Psi_{,\Theta_2} = 0.$$

Its general solution is an arbitrary function of integrals of the characteristic system of equations

$$2\Theta_2\,d\Theta_1 = \Theta_1\,d\Theta_2, \qquad dJ = 0,$$

according to which we have $\Upsilon = \Theta_2\Theta_1^{-2} = I_2I_1^{-2} = C_1$ and $J = C_2$ ($C_1, C_2 = \text{const}$). Hence, Ψ is a function of only two arguments: $\Psi = \Psi(\Upsilon, J)$. The formulas for s^{ij} in (4.2) are written in the form

$$s^{ij} = 2\beta\Theta_1^{-2}(\theta^{ij\,\prime} - \Theta_1\Upsilon g^{ij}) \qquad (\beta = \Psi_{,\Upsilon}).$$

With allowance for (2.3), (2.4), and (3.5), we divide their left and right sides by the common factor $J^{2/3}$. Using the expressions for $\sigma^{(2)ij}$ in (3.3), we obtain the equations of an isotropic hyperelastic body

$$\sigma^{ij} = \Psi_{,e_{ij}} = 2\mu(e^{ij\,\prime} - \chi g^{ij}) + \gamma\alpha^{ij}. \tag{4.3}$$

Here $\mu = \beta I_1^{-2}$, $\gamma = pJ$, $\chi = I_1\Upsilon$, and $\Upsilon = I_2I_1^{-2}$. We have $I_1 > 0$, $I_2 \geqslant 0$, $0 \leqslant \Upsilon < 2/3$, and $J > 0$. The coefficients μ and γ must satisfy the equation

$$(\mu I_1^2)_{,J} = (\gamma J^{-1})_{,\Upsilon}. \tag{4.4}$$

Thus, the coefficients in Eqs. (4.3) depend on three invariants of the strain tensor I_1, I_2, and J, whereas Ψ is a function of two arguments J and Υ. We note that in [2–6], the strain-energy density is determined as a function of three arguments.

Using (1.3), (3.2), (3.3), and (4.3), we obtain the equations relating the Cauchy stress tensor $\hat{\sigma}$ to the Almansi strain tensor $\hat{e}$:

$$\begin{gathered}\hat{\sigma}^{ij} = \mu J^{-1}(\hat{\alpha}^{in}\hat{\alpha}^j_n - \hat{\chi}\hat{\alpha}^{ij}) + p\hat{g}^{ij}, \qquad \hat{\chi} = 2I_1(\Upsilon + 1/3),\\ \hat{\alpha}^{ij} = \hat{x}^i_{,y^m}\hat{x}^j_{,y^m} = [(\hat{G} - 2\hat{E})^{-1}]^{ij}.\end{gathered} \tag{4.5}$$

Here $\hat{\alpha}^{ij}$ are the contravariant components of the tensor $\hat{\alpha} = \hat{\alpha}^{ij}\,\hat{\boldsymbol{\imath}}_i\,\hat{\boldsymbol{\imath}}_j$.

We assume that the hydrostatic pressure p depends only on the volume strain ε_V. Then $p = p(J)$, and the form of the function Ψ is considerably simplified. The latter is written in the form of two functions

$$\Psi = \Psi_1 + \Psi_2, \qquad \Psi_2 = \int p\, dJ,$$

each of which depends only on one argument: $\Psi_1 = \Psi_1(\Upsilon)$ or $\Psi_2 = \Psi_2(J)$. The coefficients μ and γ in (4.3) take the values $\mu = I_1^{-2}\Psi_{1,\Upsilon}$ and $\gamma = J\Psi_{2,J}$ and satisfy Eq. (4.4) identically.

In the case of small strains, Eqs. (4.3) and (4.5), which can be linearized with respect to strains, become the relations of Hooke's law $\sigma^{ij} = 2\mu_0 e^{ij\,\prime} + pg^{ij}$ and $p = Ke^n_n$ with two material constants, namely, the shear modulus μ_0 and the bulk modulus K, which are obtained in the limiting process $\mu \to \mu_0$ and $p/\varepsilon_V \to K$ as the strains tend to zero.

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